

LE MATEMATICHE  
Vol. LIV (1999) – Fasc. II, pp. 287–307

## SYMMETRIZATION RESULTS FOR A MULTI-EXPONENT, DEGENERATE AND ANISOTROPIC ELECTROSTATIC PROBLEM

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In this paper, we give some isoperimetric inequalities for the capacity  $c_p$  of an anisotropic configuration where each connected component has the form  $\Omega_i = \omega_i \setminus \overline{\omega'_i}$ ,  $i \in \{1, \dots, n\}$ ,  $\omega_i$  and  $\omega'_i$  are regular bounded open sets in  $\mathbb{R}^{N_i}$ , ( $N_i \geq 1$ ). The anisotropy of  $\Omega_i$  is described by a Finsler metric (or gauge function)  $\phi_i(\xi)$ ,  $\xi \in \mathbb{R}^{N_i}$  and the growth exponent  $p$  may vary with  $i$ . Using the convex symmetrization, we prove in particular that  $c_p \geq \tilde{c}_p$ , where  $\tilde{c}_p$  is the capacity of a suitable symmetrized anisotropic configuration.

### 1. Statement of the problem.

Let  $\Omega_i$  ( $i = 1, \dots, n$ ) be open sets of the form  $\Omega_i = \omega_i \setminus \overline{\omega'_i}$ , where  $\omega_i$  and  $\omega'_i$  are regular bounded open sets in  $\mathbb{R}^{N_i}$  ( $N_i \geq 1$ ) such that  $\overline{\omega'_i} \subset \omega_i$ . Let  $\gamma_i = \partial\omega_i$  and  $\gamma'_i = \partial\omega'_i$  be the respective boundaries of  $\omega_i$  and  $\omega'_i$ .

Let  $r = (r_i)$ ,  $p = (p_i)$ ,  $q = (q_i)$ ,  $i = 1 \dots, n$  be multi-exponents such that

$$(1.1) \quad 1 \leq r_i \leq \infty, \quad 1 + \frac{1}{r_i} < p_i < \infty, \quad q_i = \begin{cases} p_i & \text{if } r_i = \infty \\ \frac{r_i}{1+r_i} p_i & \text{if } r_i < \infty \end{cases}$$

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Entrato in Redazione il 9 ottobre 1999.

(hence  $1 < q_i \leq p_i$ ) and let  $a_i : \Omega_i \rightarrow \mathbb{R}$  be a (a.e.) positive function such that

$$(1.2) \quad a_i \in L^1(\Omega_i), \quad a_i^{-1} = \frac{1}{a_i} \in L^{r_i}(\Omega_i)$$

where  $L^1(\Omega_i)$  and  $L^{r_i}(\Omega_i)$  are classical Lebesgue spaces. Let

$$L_{a_i}^{p_i}(\Omega_i) = \left\{ v : \Omega_i \rightarrow \mathbb{R}, \quad \int_{\Omega_i} a_i |v|^{p_i} dx < +\infty \right\}$$

be the weighted Lebesgue space equipped with the norm

$$\|v\|_{L_{a_i}^{p_i}(\Omega_i)} = \left( \int_{\Omega_i} a_i |v|^{p_i} dx \right)^{1/p_i}$$

and let us introduce the spaces

$$\mathbb{L}^q = \{v = (v_1, \dots, v_n), \quad \forall i = 1, \dots, n, v_i \in L^{q_i}(\Omega_i)\},$$

$$\mathbb{L}_a^p = \{v = (v_1, \dots, v_n), \quad \forall i = 1, \dots, n, v_i \in L_{a_i}^{p_i}(\Omega_i)\}.$$

We equip them with the respective norms

$$\|v\|_{\mathbb{L}^q} = \sum_{i=1}^n \|v_i\|_{L^{q_i}(\Omega_i)}, \quad \|v\|_{\mathbb{L}_a^p} = \sum_{i=1}^n \|v_i\|_{L_{a_i}^{p_i}(\Omega_i)}.$$

By Hölder's inequality, (1.1) and (1.2), it is easy to check that

$$(1.3) \quad \|v\|_{\mathbb{L}^q} \leq \max_{i \in \{1, \dots, n\}} \left\{ \|a_i^{-1}\|_{L^{r_i}(\Omega_i)}^{1/p_i} \right\} \|v\|_{\mathbb{L}_a^p}$$

and it follows that  $\mathbb{L}_a^p \hookrightarrow \mathbb{L}^q$  with continuous imbedding. Moreover, let us set

$$\mathbb{W}^{1,q} = \left\{ v = (v_1, \dots, v_n), \quad \forall i = 1, \dots, n, v_i \in W^{1,q_i}(\Omega_i) \right\},$$

$$\mathbb{W}_a = \left\{ v = (v_1, \dots, v_n), \quad \forall i = 1, \dots, n, v_i \in L^{q_i}(\Omega_i), \nabla v_i \in L_{a_i}^{p_i}(\Omega_i)^{N_i} \right\},$$

where for simplicity  $\nabla$  denotes the gradient (in the sense of distributions) in any dimension.

By the previous remark,  $\mathbb{W}_a \hookrightarrow \mathbb{W}^{1,q}$  with continuous imbedding. In particular if  $v \in \mathbb{W}_a$  then  $v|_{\gamma_i}$  and  $v|_{\gamma'_i}$  are well defined and belong respectively to  $L^{q_i}(\gamma_i)$  and  $L^{q_i}(\gamma'_i)$ . Hence, we can define

$$\mathbb{H} = \left\{ v \in \mathbb{W}_a, v_1 = 1 \text{ on } \gamma'_1, v_n = 0 \text{ on } \gamma_n \text{ and } v_i|_{\gamma_i} = v_{i+1}|_{\gamma'_{i+1}} = k_i \right\}$$

(undetermined constant) for  $i = 1, \dots, n - 1$  }.

Let  $\phi_i : \mathbb{R}^{N_i} \rightarrow [0, +\infty[$  ( $i = 1, \dots, n$ ), be non negative strictly convex functions, differentiable off the origin, homogeneous in the sense

$$(1.4) \quad \forall t \in \mathbb{R}, \forall \xi \in \mathbb{R}^{N_i}, \phi_i(t\xi) = |t|\phi_i(\xi)$$

and with linear growth

$$(1.5) \quad \exists \delta > 0, \forall \xi \in \mathbb{R}^{N_i}, |\xi| \leq \phi_i(\xi) \leq \delta|\xi|,$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^{N_i}$ .

Let  $G_i : \Omega_i \times \mathbb{R}^{N_i} \rightarrow \mathbb{R}$  ( $i = 1, \dots, n$ ), be Carathéodory functions (i.e. measurable with respect to  $x$  and continuous with respect to  $\xi$ ) such that

- for almost every  $x \in \Omega_i$ ,  $G_i(x, \cdot)$  is strictly convex, homogeneous of degree  $p_i$  in the sense

$$\forall t \in \mathbb{R}, \forall \xi \in \mathbb{R}^{N_i}, G_i(x, t\xi) = |t|^{p_i} G_i(x, \xi)$$

and it admits a gradient  $g_i(x, \cdot)$ ,

- there exists  $c \geq 1$  such that for almost every  $x \in \Omega_i$  and for every  $\xi \in \mathbb{R}^{N_i}$

$$(1.6) \quad a_i(x)\phi_i(\xi)^{p_i} \leq G_i(x, \xi) \leq ca_i(x)|\xi|^{p_i}.$$

We consider the following problem

$$(1.7) \quad \inf \left\{ J(v) = \sum_{i=1}^n \frac{1}{p_i} \int_{\Omega_i} G_i(x, \nabla v_i) dx, v \in \mathbb{H} \right\},$$

the integral being finite thanks to (1.6).

For  $N_i = N$ ,  $\phi_i(\xi) = |\xi|$  and  $p_i = p$  for any  $i \in \{1, \dots, n\}$ , similar problems have been considered by V. Ferone and L. Boukrim. In an interesting paper [9], V. Ferone has given an isoperimetric inequality for the  $p$ -capacity  $c_p$  of a configuration  $\Omega = (G \setminus E) \setminus (\cup_i H_i)$ , where  $\Omega$  represents a nonhomogeneous isotropic medium,  $\partial G$  and  $\partial E$  have given potentials respectively equal to 0 and 1, and the  $H_i$  have constant unknown potentials  $K_i$ . He has shown that  $c_p \geq c_p^*$  where  $c_p^*$  is the  $p$ -capacity of a symmetrical configuration which has no interior

conductor such as  $H_i$ . In his thesis [6] (see also the short note [5]), L. Boukrim has extended and completed Ferone's result when  $\Omega$  is multiconnected and when the  $H_i$  separate the different connected components of  $\Omega$ . He proved that  $c_p \geq \bar{c}_p \geq c_p^*$ , where  $\bar{c}_p$  is the  $p$ -capacity of a symmetrized isotropic configuration (having inner conductors) and gave isoperimetric estimates for the unknown potentials  $K_i$ .

In this paper the anisotropy function  $\phi_i$ , as well the growth exponent  $p_i$ , may be different when  $i$  varies. Our purpose is to show that the generalized  $p$ -capacity of the collection of  $\Omega_i (i = 1, \dots, n)$ , denoted  $c_p$  (see section 2 below) is not smaller than the  $p$ -capacity  $\tilde{c}_p$  of a symmetrized anisotropic configuration and to give isoperimetric estimates for the unknown potentials  $K_i$ . The proof, inspired by the work of L. Boukrim, uses the notion of relative rearrangement introduced by J. Mossino and R. Temam [12] and developed in [13, 14]. But the anisotropy of  $\Omega_i$  requires other arguments related to the new notion of convex symmetrization introduced in [1].

## 2. Study of the problem.

In this section we study the existence, uniqueness and characterization of solution of problem (1.7).

**Theorem 1.** *Problem (1.7) admits a solution and only one.*

*Proof.* The proof is not quite standard in this context of degenerate problems in several domains in different dimensions and with different exponents. Let  $u^m$  be a minimizing sequence:  $u^m \in \mathbb{H}$  and  $J(u^m) \rightarrow I$ , where  $I$  denotes the infimum in (1.7). We have, due to the coerciveness condition in (1.6) together with (1.5),

$$\sum_{i=1}^n \int_{\Omega_i} a_i(x) |\nabla u_i^m|^{p_i} dx \leq J(u^m) \leq c$$

and hence  $\|\nabla u_i^m\|_{L^{p_i}(\Omega_i)^{N_i}} \leq c$  where here (and in the following) we denote by  $c$  any constant.

In particular  $\nabla u_n^m$  is bounded in  $L^{q_n}(\Omega_n)^{N_n}$ . As  $u_n^m = 0$  on  $\gamma_n$ ,  $u_n^m$  is bounded in  $W^{1,q_n}(\Omega_n)$  by Poincaré inequality. By continuity of the trace mapping (i.e.  $W^{1,q_n}(\Omega_n) \rightarrow L^{q_n}(\gamma'_n)$ ),  $k_{n-1}^m = u_n^m|_{\gamma'_n}$  is bounded in  $\mathbb{R}$ .

Now  $\nabla u_{n-1}^m$  is bounded in  $L^{q_{n-1}}(\Omega_{n-1})^{N_{n-1}}$  and  $k_{n-1}^m = u_{n-1}^m|_{\gamma_{n-1}}$  is bounded in  $\mathbb{R}$ . It follows from Poincaré inequality that  $u_{n-1}^m$  is bounded in  $W^{1,q_{n-1}}(\Omega_{n-1})$  and, just as above  $k_{n-2}^m = u_{n-1}^m|_{\gamma'_{n-1}}$  is bounded in  $\mathbb{R}$ , so

that by induction  $u_i^m$  is bounded in  $W^{1,q_i}(\Omega_i)$  (for any  $i = 1, \dots, n$ ) and  $k_i^m = u_{i|\gamma_i}^m = u_{i+1|\gamma'_{i+1}}^m$  is bounded in  $\mathbb{R}$  (for any  $i = 1, \dots, n-1$ ).

Up to an extraction of a subsequence we may suppose that for any  $i = 1, \dots, n$

$$\begin{aligned} u_i^m &\rightharpoonup u_i \text{ weakly in } W^{1,q_i}(\Omega_i), \\ u_i^m &\rightarrow u_i \text{ strongly in } L^{q_i}(\Omega_i) \text{ (by compactness),} \\ u_{i|\gamma_i}^m \text{ (resp. } u_{i|\gamma'_i}^m) &\rightarrow u_{i|\gamma_i} \text{ (resp. } u_{i|\gamma'_i}) \text{ strongly in } L^{q_i}(\Gamma_i) \text{ (resp. } L^{q_i}(\gamma'_i)), \\ \nabla u_i^m &\rightharpoonup \zeta_i \text{ weakly in } L_{a_i}^{p_i}(\Omega_i)^{N_i}, \\ k_i^m &\rightarrow k_i \text{ in } \mathbb{R}. \end{aligned}$$

As  $u^m \in \mathbb{H}$ , we get  $u_1 = 1$  on  $\gamma'_1$ ,  $u_n = 0$  on  $\gamma_n$ ,  $u_{i|\gamma_i} = u_{i+1|\gamma'_{i+1}} = k_i$  ( $i = 1, \dots, n-1$ ). As  $\nabla u_i^m \rightharpoonup \zeta_i$  weakly in  $L_{a_i}^{p_i}(\Omega_i)^{N_i}$ , we get  $\nabla u_i^m \rightharpoonup \zeta_i$  weakly in  $L^{q_i}(\Omega_i)^{N_i}$  by using the continuity of the imbedding  $L_{a_i}^{p_i}(\Omega_i) \hookrightarrow L^{q_i}(\Omega_i)$ . Since  $u_i^m \rightarrow u_i$  in  $L^{q_i}(\Omega_i)$ , it follows that  $\zeta_i = \nabla u_i \in L_{a_i}^{p_i}(\Omega_i)^{N_i}$  and  $u \in \mathbb{H}$ .

It remains to prove that  $u$  solves (1.7). We note that  $(x, \xi) \in \Omega_i \times \mathbb{R}^{N_i} \rightarrow G_i(x, \xi) \in \mathbb{R}$  is a Carathéodory function such that by (1.5) and (1.6)

$$a_i(x)|\xi|^{p_i} \leq G_i(x, \xi) \leq ca_i(x)|\xi|^{p_i}.$$

Hence the mapping  $r \rightarrow G_i(x, r)$  is continuous from  $L_{a_i}^{p_i}(\Omega_i)^{N_i}$  into  $L^1(\Omega_i)$  and the mapping  $r \rightarrow \int_{\Omega_i} G_i(x, r) dx$  is continuous from  $L_{a_i}^{p_i}(\Omega_i)^{N_i}$  into  $\mathbb{R}$ . It is also convex, so that it is lower semicontinuous for the weak topology of  $L_{a_i}^{p_i}(\Omega_i)^{N_i}$  and as  $\nabla u_i^m \rightharpoonup \nabla u_i$  in  $L_{a_i}^{p_i}(\Omega_i)^{N_i}$ ,

$$\begin{aligned} I &= \liminf \sum_{i=1}^n \frac{1}{p_i} \int_{\Omega_i} G_i(x, \nabla u_i^m) dx \geq \sum_{i=1}^n \frac{1}{p_i} \liminf \int_{\Omega_i} G_i(x, \nabla u_i^m) dx \\ &\geq \sum_{i=1}^n \frac{1}{p_i} \int_{\Omega_i} G_i(x, \nabla u_i) dx \end{aligned}$$

which proves that  $u$  solves (1.7). By the strict convexity, the gradient is the same in each  $\Omega_i$  for all solutions of (1.7) and it follows from the boundary conditions in  $\mathbb{H}$  that the solution of (1.7) is unique (and then the above convergences hold for the whole sequence  $u^m$ ). This finishes the proof of Theorem 1.  $\square$

Let  $u$  be the solution of (1.7). It is classical that  $u$  is characterized by the variational formulation:  $u \in \mathbb{H}$  and

$$(2.1) \quad 0 = \sum_{i=1}^n \frac{1}{p_i} \int_{\Omega_i} g_i(x, \nabla u_i) \cdot \nabla v_i dx, \quad \forall v \in \mathbb{H}_0,$$

with

$$\mathbb{H}_0 = \left\{ v \in \mathbb{W}_a, v_1 = 0 \text{ on } \gamma'_1, v_n = 0 \text{ on } \gamma_n \text{ and } v_{i|\gamma_i} = v_{i+1|\gamma'_{i+1}} = k_i \right. \\ \left. (\text{undetermined constant}) \text{ for } i = 1, \dots, n-1 \right\}.$$

It follows that  $u$  satisfies

$$\begin{cases} \mathcal{A}_i u_i = 0 & \text{in } \Omega_i \text{ (in the sense of distributions)} \\ u_1 = 1 & \text{on } \gamma'_1, \\ u_n = 0 & \text{on } \gamma_n, \\ u_{i|\gamma_i} = u_{i+1|\gamma'_{i+1}} = k_i & \text{(unprescribed constant) for } i = 1, \dots, n-1, \end{cases}$$

where

$$\mathcal{A}_i u_i = -\frac{1}{p_i} \operatorname{div} (g_i(x, \nabla u_i))$$

and for simplicity  $\operatorname{div}$  (resp.  $\nabla$ ) denotes the divergence (resp. gradient in any dimension  $N_i$ ).

Let  $v_i (i = 1, \dots, n)$  be the unique solution of

$$(2.2) \quad \inf \left\{ \frac{1}{p_i} \int_{\Omega_i} G_i(x, \nabla w) dx, w \in W_{a_i}(\Omega_i), w = 1 \text{ on } \gamma'_i, w = 0 \text{ on } \gamma_i \right\},$$

where

$$W_{a_i}(\Omega_i) = \{v \in L^{q_i}(\Omega_i), \nabla v \in L^{p_i}_{a_i}(\Omega_i)^{N_i}\}.$$

Then  $v_i$  is characterized by  $v_i \in W_{a_i}(\Omega_i)$ ,  $v_i = 1$  on  $\gamma'_i$ ,  $v_i = 0$  on  $\gamma_i$  and

$$(2.3) \quad \int_{\Omega_i} g_i(x, \nabla v_i) \cdot \nabla \varphi dx = 0, \forall \varphi \in W_{a_i}(\Omega_i), \varphi = 0 \text{ on } \gamma'_i \cup \gamma_i,$$

and it follows that

$$(2.4) \quad \begin{cases} \mathcal{A}_i v_i = 0 & \text{in } \Omega_i \text{ (in the sense of distributions),} \\ v_i = 1 & \text{on } \gamma'_i, \\ v_i = 0 & \text{on } \gamma_i. \end{cases}$$

Next, we prove that the solution  $u$  of (1.7) is explicit in terms of the solutions  $v_i$  ( $i = 1, \dots, n$ ) of (2.2).

**Theorem 2.** Let  $u$  be the solution of (1.7),  $k_i = u_{i+1}|_{\gamma'_{i+1}}$  and let  $v_i$  be the solution of (2.2). Let

$$c_p = \sum_{i=1}^n \int_{\Omega_i} G_i(x, \nabla u_i) dx$$

be a generalized  $p$ -capacity of the collection of  $\Omega_i (i = 1, \dots, n)$ . We have

- (a)  $c_p > 0$ ,
- (b)  $c_p = \frac{1}{p_i} \int_{\Omega_i} g_i(x, \nabla u_i) \cdot \nabla v_i dx$ , for  $i = 1, 2, \dots, n$ ,
- (c)  $\int_{\Omega_i} G_i(x, \nabla u_i) dx > 0$ ,
- (d)  $k_i \neq k_{i-1}$ ,
- (e)  $u_i = (k_{i-1} - k_i)v_i + k_i$ ,
- (f)  $\int_{\Omega_i} G_i(x, \nabla u_i) dx = (k_{i-1} - k_i)c_p$ ,
- (g)  $0 = k_n < k_{n-1} < \dots < k_{i+1} < k_i < \dots < k_1 < k_0 = 1$ ,
- (h)  $\int_{\Omega_i} G_i(x, \nabla v_i) dx = \frac{c_p}{(k_{i-1} - k_i)^{p_i-1}}$ ,
- (i)  $0 < v_i < 1$ ,  $k_i < u_i < k_{i-1}$ .

*Proof.* (a) If  $c_p = 0$  then we get from (1.5) and (1.6) that  $u_i$  is constant in each connected component  $\Omega_i$ . Using the transmission conditions (because  $u \in \mathbb{H}$ ), we obtain a contradiction.

(b) Let  $\tilde{v}^i = (\tilde{v}_1^i, \dots, \tilde{v}_n^i)$  be the function defined by

$$\tilde{v}_i^i = v_i, \quad \tilde{v}_j^i = \begin{cases} 1 & \text{if } j < i \\ 0 & \text{if } j > i. \end{cases}$$

As  $\tilde{v}^i - u \in \mathbb{H}_0$ , we get, using the variational formulation of  $u$ ,

$$0 = \sum_j \frac{1}{p_j} \int_{\Omega_j} g_j(x, \nabla u_j) \cdot \nabla (\tilde{v}_j^i - u_j) dx$$

which is equivalent to

$$c_p = \sum_j \int_{\Omega_j} G_j(x, \nabla u_j) dx = \frac{1}{p_i} \int_{\Omega_i} g_i(x, \nabla u_i) \cdot \nabla v_i dx.$$

(c) If  $\int_{\Omega_i} G_i(x, \nabla u_i) dx = 0$  then from (1.5) and (1.6), we have  $\nabla u_i = 0$  and hence  $c_p = 0$  using (b); but this contradicts (a).

(d) If  $k_i = k_{i-1}$ , then we can define  $m^i = (m_1^i, \dots, m_n^i)$  by

$$m_j^i = \begin{cases} k_i & \text{for } j = i \\ u_i & \text{otherwise} \end{cases}$$

and  $m^i$  belongs to  $\mathbb{H}$ . It follows from (c) that

$$\sum_j \int_{\Omega_j} \frac{1}{p_j} G_j(x, \nabla m_j^i) dx < \sum_j \int_{\Omega_j} \frac{1}{p_j} G_j(x, \nabla u_j) dx$$

which contradicts the minimality property of  $u$ .

(e) Following (d), one can define  $w_i = \frac{u_i - k_i}{k_{i-1} - k_i}$ . It is easy to check (from  $\mathcal{A}_i u_i = 0$ ), that  $\mathcal{A}_i w_i = 0$ ,  $w_i = 1$  on  $\gamma'_i$ ,  $w_i = 0$  on  $\gamma_i$ . The functions  $v_i$  and  $w_i$  satisfy the same equation which has a unique solution. It follows that  $w_i = v_i$ .

(f) It is sufficient to replace  $v_i$  by  $\frac{u_i - k_i}{k_{i-1} - k_i}$  in (b).

(g) Clear from (a), (c) and (f).

(h) Replace  $u_i$  by  $(k_{i-1} - k_i)v_i + k_i$  in (b).

(i) Using convenient test functions in (2.3), it is easy to prove that  $0 < v_i < 1$  and then (e) gives  $k_i < u_i < k_{i-1}$ .  $\square$

**Remark 1.** From (h) of Theorem 2,  $\sum_{i=1}^n (k_{i-1} - k_i) = 1$  and  $k_i = 1 - \sum_{j=1}^i (k_{j-1} - k_j)$ , we get

$$1 = \sum_{i=1}^n \left( \frac{c_p}{\int_{\Omega_i} G_i(x, \nabla v_i) dx} \right)^{\frac{1}{p_i-1}}$$

and

$$k_i = 1 - \sum_{j=1}^i \left( \frac{c_p}{\int_{\Omega_j} G_j(x, \nabla v_j) dx} \right)^{\frac{1}{p_j-1}}$$

**Remark 2.** If Green's formula is valid, then we have from  $\mathcal{A}_i u_i = 0$  in  $\Omega_i$  and from (b) of Theorem 2 that for all  $i = 1, \dots, n$

$$c_p = - \int_{\gamma'_i} \frac{\partial u_i}{\partial v^{\mathcal{A}_i}} d\gamma = - \int_{\gamma_i} \frac{\partial u_i}{\partial v^{\mathcal{A}_i}} d\gamma$$



where

$$\frac{\partial u_i}{\partial \nu^{\mathcal{A}_i}} = \frac{1}{p_i} g_i(x, \nabla u_i) \cdot \nu$$

and for simplicity  $\nu$  denotes the outer normal to  $\Omega_i$  on  $\gamma_i$  as well as the inner normal to  $\Omega_i$  on  $\gamma'_i$ .

### 3. Main inequalities.

Let us recall some notions of (unidimensional and relative) rearrangement (see for example [3], [8], [11], [12], [13], [14]). In this paper, we use only the Lebesgue measure on  $\mathbb{R}^N$  (for different values of  $N$ ). For a measurable set  $E$  in  $\mathbb{R}^N$ , let  $|E|$  be its measure. Let  $u$  be a measurable function from  $E$  into  $\mathbb{R}$ . The (unidimensional) decreasing rearrangement  $u_*$  of  $u$  is defined on  $\overline{E}^* = [0, |E|]$  by  $u_*(|E|) = \operatorname{ess\,inf}_E u$  and for  $s < |E|$ ,  $u_*(s) = \inf\{\theta \in \mathbb{R}, |u > \theta| \leq s\}$  where  $|u > \theta| = |\{x \in E : u(x) > \theta\}|$ ; the increasing rearrangement of  $u$ , denoted  $u^*$ , is then  $u^*(s) = u_*(|E| - s)$ . The functions  $u$ ,  $u_*$  and  $u^*$  satisfy  $|u > \theta| = |u_* > \theta| = |u^* > \theta|$ .

For  $v \in L^1(E)$  and  $u : E \rightarrow \mathbb{R}$  measurable, we define the function  $\mathcal{W}$  on  $\overline{E}^*$  by

$$\mathcal{W}(s) = \begin{cases} \int_{u > u_*(s)} v(x) dx & \text{if } |u - u_*(s)| = 0, \\ \int_{u > u_*(s)} v(x) dx + \int_0^{s - |u > u_*(s)|} (v|_{P(s)})_*(\sigma) d\sigma & \text{otherwise,} \end{cases}$$

where  $(v|_{P(s)})_*$  is the decreasing rearrangement of  $v$  restricted to  $P(s) = \{x \in E : u(x) = u_*(s)\}$ . The integrable function  $\frac{d\mathcal{W}}{ds}$  is called (according to [12], [13], [14]) the relative rearrangement of  $v$  with respect to  $u$  and is denoted  $v_{*u}$ .

We recall also some facts about the function  $\phi_i$  defined in section 1. As it has been said earlier, the function  $\phi_i : \mathbb{R}^{N_i} \rightarrow [0, +\infty[$  is strictly convex, homogeneous of degree one, with linear growth and differentiable off the origin.

Let

$$B_{\phi_i} = \{\xi \in \mathbb{R}^{N_i}; \phi_i(\xi) \leq 1\}$$

be the unit ball of  $\mathbb{R}^{N_i}$  relative to  $\phi_i$ . It follows from the definition of  $\phi_i$  that the ball  $B_{\phi_i}$  (the so-called Wulff shape relative to  $\phi_i$ ) is bounded, convex and symmetric with respect to the origin.

We denote by  $\phi_i^0 : \mathbb{R}^{N_i} \rightarrow [0, +\infty[$  the dual function of  $\phi_i$  defined by

$$\phi_i^0(\xi^*) = \sup\{\xi^* \cdot \xi; \xi \in B_{\phi_i}\} = \sup_{\xi \neq 0} \frac{\xi^* \cdot \xi}{\phi_i(\xi)}, \quad \forall \xi^* \in \mathbb{R}^{N_i}.$$

One can check that  $\phi_i^0$  is also a convex function and satisfies the properties (1.4) and  $\frac{1}{\delta}|\xi^*| \leq \phi_i^0(\xi^*) \leq |\xi^*|$  (see for example [15]). In the sequel, we assume that the dual function  $\phi_i^0$  is strictly convex and differentiable everywhere but in the origin. The corresponding unit ball  $B_{\phi_i^0}$  is known as Frank diagram. One can also prove from (1.4) the following useful properties of the functions  $\phi_i$  and  $\phi_i^0$  (see for example [4]). Let  $\xi \in \mathbb{R}^{N_i} \setminus \{0\}$  and let  $t \neq 0$ , then

$$(3.1) \quad \nabla \phi_i(t\xi) = \frac{t}{|t|} \nabla \phi_i(\xi), \quad \nabla \phi_i^0(t\xi) = \frac{t}{|t|} \nabla \phi_i^0(\xi)$$

$$(3.2) \quad \phi_i(\xi) = \nabla \phi_i(\xi) \cdot \xi, \quad \phi_i^0(\xi) = \nabla \phi_i^0(\xi) \cdot \xi$$

$$(3.3) \quad 1 = \phi_i(\nabla \phi_i^0(\xi)) = \phi_i^0(\nabla \phi_i(\xi))$$

$$(3.4) \quad \xi = \phi_i^0(\xi) \nabla \phi_i(\nabla \phi_i^0(\xi)) = \phi_i(\xi) \nabla \phi_i^0(\nabla \phi_i(\xi)).$$

All the isoperimetric inequalities of this section are consequences of the following theorem.

**Theorem 3.** *Let  $i \in \{1, \dots, n\}$ . Let  $\alpha_i$  be the Lebesgue measure of the unit ball (i.e. Frank diagram)  $B_{\phi_i^0} = \{\xi \in \mathbb{R}^{N_i}; \phi_i^0(\xi) \leq 1\}$  in  $\mathbb{R}^{N_i}$ . Let  $p'_i$  be such that  $\frac{1}{p_i} + \frac{1}{p'_i} = 1$  and let  $v_i$  be the unique solution of (2.2). Then for all  $t, t'$  such that  $0 \leq t \leq t' \leq 1$ , we have*

$$t' - t \leq N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i} \cdot \int_{|v_i > t'|}^{|v_i > t|} (|\omega'_i| + \sigma)^{\frac{p'_i}{N_i} - p'_i} (a_i^*)^{-\frac{p'_i}{p_i}} (\sigma - |v_i > t'|) d\sigma.$$

*Proof.* For  $\theta \in ]0, 1[$ , let us set

$$z_i = \theta - (v_i - \theta)_- = \begin{cases} v_i & \text{if } v_i \leq \theta \\ \theta & \text{if } v_i > \theta. \end{cases}$$

Then the function  $\varphi = z_i - \theta v_i$  satisfies the conditions  $\varphi \in W_{a_i}(\Omega_i)$ ,  $\varphi = 0$  on  $\gamma'_i \cup \gamma_{i+1}$ . In consequence, we have using (2.3)

$$0 = \int_{\Omega_i} g_i(x, \nabla v_i) \cdot \nabla(z_i - \theta v_i) dx.$$

Hence

$$\int_{v_i \leq \theta} G_i(x, \nabla v_i) dx = \theta \int_{\Omega_i} G_i(x, \nabla v_i) dx$$

and then

$$(3.5) \quad \frac{d}{d\theta} \int_{v_i > \theta} G_i(x, \nabla v_i) dx = - \int_{\Omega_i} G_i(x, \nabla v_i) dx.$$

Moreover, by using (1.6), (1.2) and Hölder's inequality, we have for  $h > 0$ ,

$$\begin{aligned} \frac{1}{h} \int_{\theta < v_i \leq \theta+h} \phi_i(\nabla v_i) dx &\leq \left( \frac{1}{h} \int_{\theta < v_i \leq \theta+h} a_i^{-p'_i/p_i} dx \right)^{1/p'_i} \\ &\quad \cdot \left( \frac{1}{h} \int_{\theta < v_i \leq \theta+h} G_i(x, \nabla v_i) dx \right)^{1/p_i} \end{aligned}$$

and letting  $h$  tend to 0, we get (thanks to (3.5))

$$\begin{aligned} -\frac{d}{d\theta} \int_{v_i > \theta} \phi_i(\nabla v_i) dx &\leq \left( -\frac{d}{d\theta} \int_{v_i > \theta} a_i^{-p'_i/p_i} dx \right)^{1/p'_i} \\ &\quad \cdot \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{1/p_i} \end{aligned}$$

By using the following formula of derivation (see [14])

$$\frac{d}{d\theta} \int_{v_i > \theta} a_i^{-p'_i/p_i} dx = \mathcal{W}'(v_i(\theta)) v'_i(\theta)$$

where  $v_i(\theta) = |v_i > \theta|$  and  $\mathcal{W}' = (a_i^{-p'_i/p_i})_{*v_i}$  is the relative rearrangement of  $a_i^{-p'_i/p_i}$  with respect to  $v_i$  it comes

$$(3.6) \quad -\frac{d}{d\theta} \int_{v_i > \theta} \phi_i(\nabla v_i) dx \leq \left( -\mathcal{W}'(v_i(\theta)) v'_i(\theta) \right)^{1/p'_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{1/p_i}$$

Let  $P_{\phi_i, \Omega_i}(\{v_i > \theta\})$  be the generalized perimeter relative to  $\phi_i$  and  $\Omega_i$  of the set  $\{x \in \Omega_i, v_i(x) > \theta\}$  defined in [2] by

$$P_{\phi_i, \Omega_i}(\{v_i > \theta\}) = \sup \left\{ \int_{v_i > \theta} \operatorname{div}(\sigma) dx; \sigma \in C_0^1(\Omega_i, \mathbb{R}^{N_i}), \phi_i^0(\sigma) \leq 1 \right\}.$$

The following two results hold (see [1]):

$$(3.7) \quad -\frac{d}{d\theta} \int_{v_i > \theta} \phi_i(\nabla v_i) dx = P_{\phi_i, \Omega_i}(\{v_i > \theta\}),$$

$$(3.8) \quad P_{\phi_i, \Omega_i}(\{v_i > \theta\}) \leq N_i \alpha_i^{1/N_i} (|\omega'_i| + v_i(\theta))^{1 - \frac{1}{N_i}}$$

Let's note that for  $\phi_i(\xi) = |\xi|$ , the result (3.7) is nothing else the Fleming-Rishel formula (see [10]) and the corresponding inequality (3.8) is known as the isoperimetric inequality for the perimeter of De Giorgi (see [7]).

Now, using (3.6), (3.7) and (3.8), we get

$$1 \leq N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i} \cdot (|\omega'_i| + v_i(\theta))^{\frac{p'_i}{N_i} - p'_i} \mathcal{W}'(v_i(\theta))(-v'_i(\theta)).$$

By integrating between  $t$  and  $t'$ ,

$$\begin{aligned} t' - t &\leq N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i} \cdot \int_0^{|\Omega_i|} \chi[v_i(t'), (t)](\sigma)(|\omega'_i| + \sigma)^{\frac{p'_i}{N_i} - p'_i} (a_i^{-p'_i/p_i})_{*v_i}(\sigma) d\sigma \\ &\leq N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i} \cdot \int_0^{|\Omega_i|} \left( \chi[v_i(t'), v_i(t)](\cdot)(|\omega'_i| + \cdot)^{\frac{p'_i}{N_i} - p'_i} \right)_*(\sigma) (a_i^{-p'_i/p_i})_*(\sigma) d\sigma \end{aligned}$$

(for this latest inequality, see Theorem 3 in [13])

$$= N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i}.$$

$$\cdot \int_0^{|\Omega_i|} \chi[0, v_i(t) - v_i(t')](\sigma)(|\omega'_i| + v_i(t') + \sigma)^{\frac{p'_i}{N_i} - p'_i} (a_i^*)^{-\frac{p'_i}{p_i}}(\sigma) d\sigma$$

(using the properties of the (unidimensional) decreasing rearrangement)

$$= N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i}.$$

$$\cdot \int_{v_i(t')}^{v_i(t)} (|\omega'_i| + \sigma)^{\frac{p'_i}{N_i} - p'_i} (a_i^*)^{-\frac{p'_i}{p_i}}(\sigma - v_i(t')) d\sigma.$$

Therefore

$$t' - t \leq N_i^{-p'_i} \alpha_i^{-p'_i/N_i} \left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i}.$$

$$\cdot \int_{|v_i > t'|}^{|v_i > t|} (|\omega'_i| + \sigma)^{\frac{p'_i}{N_i} - p'_i} (a_i^*)^{-\frac{p'_i}{p_i}}(\sigma - |v_i > t'|) d\sigma$$

for all  $t, t'$  such that  $0 \leq t \leq t' \leq 1$ .  $\square$

Making  $t = 0$  and  $t' = 1$  in Theorem 3, we obtain

**Corollary 1.** Let  $i \in \{1, \dots, n\}$ ,  $f_i(\sigma) = (|\omega'_i| + \sigma)^{\frac{p'_i}{N_i} - p'_i} (a_i^*)^{-\frac{p'_i}{p_i}}(\sigma)$  for  $\sigma \in [0, |\Omega_i|]$  and  $I_i = \int_0^{|\Omega_i|} f_i(\sigma) d\sigma$ . We have

$$\left( \int_{\Omega_i} G_i(x, \nabla v_i) dx \right)^{p'_i/p_i} \geq \frac{N_i^{p'_i} \alpha_i^{\frac{p'_i}{N_i}}}{I_i}.$$

Now we are able to state our main results of this section.

**Theorem 4.** Let  $S$  be the unique positive solution of  $\sum_{i=1}^n \frac{I_i S^{p'_i-1}}{N_i^{p'_i} \alpha_i^{p'_i/N_i}} = 1$ .

We have  $c_p \geq S$ . Moreover for any  $s \in [0, |\Omega_i|]$ ,

$$(3.9) \quad (c_p)^{-\frac{p'_i}{p_i}} (k_{i-1} - u_{i*}(s)) \leq N_i^{-p'_i} \alpha_i^{-\frac{p'_i}{N_i}} \int_0^s f_i(\sigma) d\sigma,$$

which gives for  $s = |\Omega_i|$

$$(3.10) \quad (c_p)^{-\frac{p'_i}{p_i}} (k_{i-1} - k_i) \leq N_i^{-p'_i} \alpha_i^{-\frac{p'_i}{N_i}} I_i.$$

*Proof.* (a) From Remark 1 and Corollary 1, we have

$$1 = \sum_{i=1}^n \left( \frac{c_p}{\int_{\Omega_i} G_i(x, \nabla v_i) dx} \right)^{\frac{1}{p_i-1}} \leq \sum_{i=1}^n \frac{I_i c_p^{p'_i-1}}{N_i^{p'_i} \alpha_i^{p'_i/N_i}}.$$

As the last expression is (strictly) increasing in  $c_p$ , we get  $c_p \geq S$ .

(b) From Theorem 2 and Theorem 3, we deduce that for all  $t$  such that  $k_i \leq t \leq k_{i-1}$

$$k_{i-1} - t \leq N_i^{-p'_i} \alpha_i^{-\frac{p'_i}{N_i}} (c_p)^{\frac{p'_i}{p_i}} \int_0^{|u_i > t|} f_i(\sigma) d\sigma.$$

Making  $t = u_{i*}(s)$ ,  $s$  in  $[0, |\Omega_i|]$  and noticing that  $|u_i > u_{i*}(s)| \leq s$ , we obtain (3.9).

#### 4. Symmetrized problem and isoperimetric inequalities.

We begin by recalling the notion of convex symmetrization introduced in the paper of A. Alvino, V. Ferone, P. L. Lions and Trombetti [1].

For  $i = 1, \dots, n$ , let  $\phi_i : \mathbb{R}^{N_i} \rightarrow [0, +\infty[$  be a strictly convex function, differen differentiable off the origin, satisfying (1.4) and (1.5). Let  $\phi_i^0$  be its dual and  $B_{\phi_i^0} = \{\xi \in \mathbb{R}^{N_i}; \phi_i^0(\xi) \leq 1\}$  be the unit ball of  $\mathbb{R}^{N_i}$  relative to  $\phi_i^0$  (i.e. the Frank diagram relative to  $\phi_i$ ) with Lebesgue measure  $\alpha_i$ . Moreover, we assume that the dual function  $\phi_i^0$  is strictly convex and differentiable everywhere but in the origin.

Let  $E$  be a measurable set in  $\mathbb{R}^{N_i}$  and let  $u$  be a measurable function from  $E$  into  $\mathbb{R}$ . Let  $\tilde{E}_i$  be the set homothetic to the Frank diagram  $B_{\phi_i^0}$  such that  $|\tilde{E}_i| = |E|$ . Note that both  $E$  and  $\tilde{E}_i$  are subsets of  $\mathbb{R}^{N_i}$ .

The convex symmetrization (or convex symmetric decreasing rearrangement) relative to  $\phi_i^0$  of  $u$ , denoted by  $u_i^c$  is defined on  $\tilde{E}_i$  by

$$u_i^c(x) = u_*(\alpha_i(\phi_i^0(x))^{N_i}); \quad x \in \tilde{E}_i.$$

The function  $u$  and  $u_i^c$  are equimeasurable. The level sets of  $u_i^c$ ,  $\{x \in \tilde{E}_i; u_i^c(x) > t\}$ , are homothetic to  $B_{\phi_i^0}$  and have the same measure as  $\{x \in E; u(x) > t\}$ . Indeed, the convex symmetrization coincides with the Schwarz symmetrization (or spherically symmetric increasing rearrangement) when  $\phi_i(\xi) = |\xi|$ .

Now let  $\tilde{\omega}_i$  (resp.  $\tilde{\omega}'_i$ ) be the set of  $\mathbb{R}^{N_i}$ , homothetic to the ball  $B_{\phi_i^0}$  such that  $|\tilde{\omega}_i| = |\omega_i|$  (resp.  $|\tilde{\omega}'_i| = |\omega'_i|$ ). The sets  $\tilde{\omega}_i$  and  $\tilde{\omega}'_i$  are bounded, convex, symmetric with respect to the origin and homothetic. Moreover  $\overline{\tilde{\omega}'_i} \subset \tilde{\omega}_i$ . Let  $A_i = \tilde{\omega}_i \setminus \overline{\tilde{\omega}'_i}$ ,  $\tilde{\gamma}_i = \partial\tilde{\omega}_i$ ,  $\tilde{\gamma}'_i = \partial\tilde{\omega}'_i$ . Let  $\mu$  be the normal to  $\tilde{\gamma}_i$  pointing outside  $A_i$  or the normal to  $\tilde{\gamma}'_i$  pointing inside  $A_i$ .

Let  $\tilde{a}_i : A_i \rightarrow \mathbb{R}$  be the function defined by

$$\tilde{a}_i(x) = a_i^*(\alpha_i(\phi_i^0(x))^{N_i} - |\tilde{\omega}'_i|)$$

where  $a_i^*$  is the increasing rearrangement of  $a_i$ . As the function  $a_i$ , the function  $\tilde{a}_i$  also satisfies (1.2) (with  $A_i$  instead of  $\Omega_i$ ).

We begin by the explicit resolution of the symmetrized problem corresponding to (2.4) in  $A_i$ .

**Proposition 1.** For  $i \in \{1, \dots, n\}$ , let  $\mathcal{B}_i$  and  $\frac{\partial}{\partial \mu \mathcal{B}_i}$  be the operators defined by

$$\begin{aligned} \mathcal{B}_i V &= -\operatorname{div}[\tilde{a}_i \phi_i(\nabla V)^{p_i-1} \nabla \phi_i(\nabla V)], \\ \frac{\partial V}{\partial \mu \mathcal{B}_i} &= \tilde{a}_i \phi_i(\nabla V)^{p_i-1} \nabla \phi_i(\nabla V) \cdot \mu \end{aligned}$$

and let  $V_i$  be the solution of the following problem

$$(4.1) \quad \begin{cases} \mathcal{B}_i V_i = 0 & \text{in } A_i, \\ V_i = 0 & \text{on } \tilde{\gamma}_i, \\ V_i = 1 & \text{on } \tilde{\gamma}'_i. \end{cases}$$

We have, with  $f_i$  and  $I_i$  defined in Corollary 1,

$$(a) \quad V_i(x) = \frac{1}{I_i} \int_{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|}^{|\Omega_i|} f_i(\sigma) d\sigma,$$

$$(b) \quad - \int_{\tilde{\gamma}'_i} \frac{\partial V_i}{\partial \mu \mathcal{B}_i} d\gamma = N_i^{p_i} \alpha_i^{p_i/N_i} I_i^{1-p_i}.$$

*Proof.* (a) With  $x \in A_i$  and  $r = \phi_i^0(x)$ , we obtain  $\nabla V_i = \frac{dV_i}{dr} \nabla \phi_i^0$ . Using the properties of  $\phi_i$  and  $\phi_i^0$ , we get for  $x \in A_i$ ,

$$\phi_i(\nabla V_i(x)) = \phi_i\left(\frac{dV_i}{dr} \nabla \phi_i^0(x)\right) = \left|\frac{dV_i}{dr}\right| \quad \text{by (1.4) and (3.3),}$$

$$\nabla \phi_i(\nabla V_i(x)) = \nabla \phi_i\left(\frac{dV_i}{dr} \nabla \phi_i^0(x)\right) = \frac{dV_i}{dr} \left| \frac{dV_i}{dr} \right|^{-1} \frac{x}{\phi_i^0(x)} \text{ by (3.1) and (3.4).}$$

Therefore the operator  $\mathcal{B}_i$  can be rewritten as

$$\begin{aligned} -\mathcal{B}_i V_i &= a_i^*(\alpha_i r^{N_i} - |\omega'_i|) \left| \frac{dV_i}{dr} \right|^{p_i-2} \frac{dV_i}{dr} \frac{1}{r} \left( N_i - \frac{1}{r} \nabla \phi_i^0(x) \cdot x \right) \\ &+ (p_i - 1) a_i^*(\alpha_i r^{N_i} - |\omega'_i|) \left| \frac{dV_i}{dr} \right|^{p_i-2} \frac{d^2 V_i}{dr^2} \frac{1}{r} \nabla \phi_i^0(x) \cdot x \\ &+ \frac{d}{dr} a_i^*(\alpha_i r^{N_i} - |\omega'_i|) \left| \frac{dV_i}{dr} \right|^{p_i-2} \frac{dV_i}{dr} \frac{1}{r} \nabla \phi_i^0(x) \cdot x. \end{aligned}$$

We see by (3.2) that

$$\nabla \phi_i^0(x) \cdot x = \phi_i^0(x) = r$$

and finally

$$\begin{aligned} -\mathcal{B}_i V_i &= \left| \frac{dV_i}{dr} \right|^{p_i-2} \left[ \frac{d}{dr} a_i^*(\alpha_i r^{N_i} - |\omega'_i|) \frac{dV_i}{dr} + \right. \\ &\left. + a_i^*(\alpha_i r^{N_i} - |\omega'_i|) \left( (p_i - 1) \frac{d^2 V_i}{dr^2} + \frac{N_i - 1}{r} \frac{dV_i}{dr} \right) \right]. \end{aligned}$$

Therefore  $\mathcal{B}_i V_i = 0$  is equivalent to

$$\frac{dV_i}{dr} = k r^{\frac{N_i-1}{1-p_i}} (a_i^*)^{-\frac{p'_i}{p_i}} (\alpha_i r^{N_i} - |\omega'_i|)$$

where  $k$  is a constant. Hence, since  $V_i = 0$  on  $\tilde{\gamma}_i$  and  $V_i = 1$  on  $\tilde{\gamma}'_i$ , we deduce that

$$V_i(x) = \frac{1}{I_i} \int_{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|}^{|\Omega_i|} f_i(\sigma) d\sigma$$

for all  $x \in A_i$ .

**(b)** We have, from earlier computations, for  $x$  in  $\tilde{\gamma}'_i$ ,

$$\frac{\partial V_i}{\partial \mu^{\mathcal{B}_i}} = a_i^*(\alpha_i r^{N_i} - |\omega'_i|) \left| \frac{dV_i}{dr} \right|^{p_i-2} \frac{dV_i}{dr} \frac{x \cdot \mu}{\phi_i^0(x)} =$$



$$= -N_i^{p_i-1} \alpha_i^{\frac{p_i}{N_i}-1} r^{-N_i} (I_i)^{1-p_i} x \cdot \mu$$

and then,

$$- \int_{\tilde{\gamma}'_i} \frac{\partial V_i}{\partial \mu^{\mathcal{B}_i}} d\gamma = N_i^{p_i-1} \alpha_i^{\frac{p_i}{N_i}-1} r^{-N_i} (I_i)^{1-p_i} \int_{\tilde{\gamma}'_i} x \cdot \mu d\gamma = N_i^{p_i} \alpha_i^{p_i/N_i} I_i^{1-p_i}$$

because  $\int_{\tilde{\gamma}'_i} x \cdot \mu d\gamma = N_i |\tilde{\omega}'_i| = N_i |\omega'_i|$  and if  $x \in \tilde{\gamma}'_i$  we have  $\alpha_i \phi_i^0(x)^{N_i} = |\tilde{\omega}'_i| = |\omega'_i|$ .

This ends the proof of Proposition 1.  $\square$

**Remark 3.** Similar computations show that one has also

$$\int_{\tilde{\gamma}_i} \frac{\partial V_i}{\partial \mu^{\mathcal{B}_i}} d\gamma = \int_{\tilde{\gamma}'_i} \frac{\partial V_i}{\partial \mu^{\mathcal{B}_i}} d\gamma,$$

$$\int_{A_i} \tilde{a}_i \phi_i (\nabla V_i)^{p_i} dx = N_i^{p_i} \alpha_i^{p_i/N_i} I_i^{1-p_i} = - \int_{\tilde{\gamma}'_i} \frac{\partial V_i}{\partial \mu^{\mathcal{B}_i}} d\gamma$$

that is Green's formula is valid.

We consider the symmetrized problem defined as follows

$$(4.2) \quad \inf \left\{ \sum_{i=1}^n \frac{1}{p_i} \int_{A_i} Q_i(x, \nabla V_i(x)) dx, \quad V \in \tilde{\mathbb{H}} \right\}$$

where

$$Q_i(x, \xi) = \tilde{a}_i(x) \phi_i(\xi)^{p_i}$$

and

$$\tilde{\mathbb{H}} = \{ V \in \mathbb{W}_{\tilde{a}}, V_1 = 1 \text{ on } \tilde{\gamma}'_1, V_n = 0 \text{ on } \tilde{\gamma}_n \text{ and } V_{i|\tilde{\gamma}_i} = V_{i+1|\tilde{\gamma}'_{i+1}} = K_i \\ \text{(undetermined constant) for } i = 1, \dots, n-1 \}.$$

**Remark 4.** It follows from Theorem 1 that the symmetrized problem (4.2) admits too one solution and only one.

Let us denote by  $U$  the solution of the symmetrized problem (4.2). Let  $K_i$  be the common value of  $U_i$  on  $\tilde{\gamma}_i$  and  $U_{i+1}$  on  $\tilde{\gamma}'_{i+1}$  ( $i = 1, \dots, n-1$ ). Let  $\tilde{c}_p = \sum_{i=1}^n \int_{A_i} Q_i(x, \nabla U_i(x)) dx$  be the generalized  $p$ -capacity of the collection of  $A_i$  ( $i = 1, \dots, n$ ). It follows from Theorem 2 applied with  $\tilde{c}_p, U_i, K_i, V_i$  instead of  $c_p, u_i, k_i, v_i$  and Remark 3 that Green's formula is also valid for  $U_i$ , so that  $U_1, \dots, U_n$  satisfy:

$$(4.3) \quad \begin{cases} \mathcal{B}_i U_i = 0 & \text{in } A_i, \\ U_i = 1 & \text{on } \tilde{\gamma}'_1, \\ U_n = 0 & \text{on } \tilde{\gamma}_n, \\ U_i|_{\tilde{\gamma}_i} = U_{i+1}|_{\tilde{\gamma}'_{i+1}} = K_i & \text{(unprescribed constant)} \\ & \text{for } (i = 1, \dots, n-1), \\ \int_{\tilde{\gamma}_i} \frac{\partial U_i}{\partial \mu^{\mathcal{B}_i}} d\gamma = \int_{\tilde{\gamma}'_i} \frac{\partial U_i}{\partial \mu^{\mathcal{B}_i}} d\gamma & \text{is independent of } i = 1, \dots, n. \end{cases}$$

The symmetrized problem can be solved explicitly:

**Theorem 5.** (explicit resolution of the symmetrized problem). *Let  $U$  be the solution of (4.2),  $K_i = U_i|_{\tilde{\gamma}_i} = U_{i+1}|_{\tilde{\gamma}'_{i+1}}$  ( $i = 1, \dots, n-1$ ). Let  $\tilde{c}_p = \sum_{i=1}^n \int_{A_i} Q_i(x, \nabla U_i(x)) dx$ . Then the values  $\tilde{c}_p, K_i$  and  $U_i$  are given respectively by*

$$(1) \quad 1 = \sum_{i=1}^n \frac{I_i(\tilde{c}_p)^{p'_i-1}}{N_i^{p'_i} \alpha_i^{p'_i/N_i}}$$

$$(2) \quad K_i = 1 - \sum_{j=1}^i \frac{I_j(\tilde{c}_p)^{p'_j-1}}{N_j^{p'_j} \alpha_j^{p'_j/N_j}}$$

and for  $i \in \{1, \dots, n\}, x \in A_i$

$$(3) \quad U_i(x) = K_{i-1} - N_i^{-p'_i} \alpha_i^{-p'_i/N_i} (\tilde{c}_p)^{\frac{p'_i}{p_i}} \int_0^{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|} f_i(\sigma) d\sigma.$$

(As already mentioned there exists a unique  $\tilde{c}_p > 0$  solution of (1).)

*Proof.* Using Theorem 2, Proposition 1 and Remarks 1 and 3, we have

$$1 = \sum_{i=1}^n \left( \frac{\tilde{c}_p}{\int_{A_i} \tilde{\alpha}_i \phi_i (\nabla V_i)^{p_i} dx} \right)^{\frac{1}{p_i-1}} = \sum_{i=1}^n \frac{I_i(\tilde{c}_p)^{p'_i-1}}{N_i^{p'_i} \alpha_i^{p'_i/N_i}},$$

$$1 - K_i = \sum_{j=1}^i \left( \frac{\tilde{c}_p}{\int_{A_j} \tilde{a}_j \phi_j (\nabla V_j)^{p_j} dx} \right)^{\frac{1}{p_j-1}} = \sum_{j=1}^i \frac{I_j(\tilde{c}_p)^{p_j'-1}}{N_j^{p_j'} \alpha_j^{p_j'/N_j}},$$

and finally for  $x \in A_i (i = 1, \dots, n)$ ,

$$\begin{aligned} U_i(x) &= (K_{i-1} - K_i)V_i(x) + K_i = K_{i-1} - (K_{i-1} - K_i)(1 - V_i(x)) \\ &= K_{i-1} - \frac{I_i(\tilde{c}_p)^{p_i'-1}}{N_i^{p_i'} \alpha_i^{p_i'/N_i}} \left( 1 - \frac{1}{I_i} \int_{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|}^{|\Omega_i|} f_i(\sigma) d\sigma \right) \\ &= K_{i-1} - N_i^{-p_i'} \alpha_i^{-p_i'/N_i} (\tilde{c}_p)^{\frac{p_i'}{p_i}} \int_0^{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|} f_i(\sigma) d\sigma. \end{aligned}$$

□

**Remark 5.** For the symmetrized problem, it follows from Theorem 5 that (3.9) becomes an equality. Actually for  $x \in \bar{A}_i$ ,  $s = \alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|$  belongs to  $[0, |\Omega_i|]$  and  $U_{i_*}(s) = U_i(x)$ .

To summarize, the following theorem says that the inequalities in Theorem 4 are all isoperimetric.

**Theorem 6.** (isoperimetric inequalities).

- a)  $c_p \geq \tilde{c}_p$ ,
- b)  $(c_p)^{-\frac{p_i'}{p_i}} (k_{i-1} - u_{i_*}(\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|)) \leq (\tilde{c}_p)^{-\frac{p_i'}{p_i}} (K_{i-1} - U_i(x))$  for  $x \in \bar{A}_i$ ,
- c)  $(c_p)^{-\frac{p_i'}{p_i}} (k_{i-1} - k_i) \leq (\tilde{c}_p)^{-\frac{p_i'}{p_i}} (K_{i-1} - K_i)$ .

*Proof.* a) is already proved (see Theorem 4 and (1) in Theorem 5).

- b) Let  $i \in \{1, \dots, n\}$ . For  $x \in \bar{A}_i$  and  $s = \alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|$ , we have by (3.9) of Theorem 4 and (3) of Theorem 5,

$$\begin{aligned} (c_p)^{-\frac{p_i'}{p_i}} (k_{i-1} - u_{i_*}(s)) &\leq N_i^{-p_i'} \alpha_i^{-\frac{p_i'}{N_i}} \int_0^{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|} f_i(\sigma) d\sigma \\ &= (\tilde{c}_p)^{-\frac{p_i'}{p_i}} (K_{i-1} - U_i(x)). \end{aligned}$$

Finally, (c) is a particular case of (b) with  $x \in \tilde{\gamma}_i$ . □

**Acknowledgments.** The author wishes to thank J. Mossino for her assistance during the realization of this work.

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